Comparison of Analytical Models of Transformer Leakage Inductance: Accuracy Versus Computational Effort

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Abstract—A fast and accurate model of the transformer leakage inductance is crucial for optimization-based design of galvanically isolated converters. Analytical models are fastly executable, and therefore, especially suitable for such optimizations. This article compares several analytical leakage inductance per unit length models with respect to the accuracy and computational effort. The considered models are applicable to E-Core and U-Core transformers. 2D FEM simulations are used as a benchmark to evaluate the model accuracy, whereas the computation time is extracted as an indicator for computational effort. Six different transformer prototypes provide the geometries for the comparison. Based on the conducted comparisons, Roth’s model is the most accurate. Rogowski’s model is the fastest. Margueron’s model is the most versatile as it takes the finite permeability of the core into account. The conducted comparisons lay the foundation for accurate and fast Double-2D modeling of the transformer leakage inductance as it is executed for the two main cross sections of E-core and U-core transformers: inside the transformer window, and outside the transformer window. This article is accompanied by a supplementary document summarizing the equations of Roth’s and Margueron’s model.

Index Terms—Analytical leakage inductance modeling, Double-2D, galvanically isolated converters, optimization, solid-state transformers, transformer, U-core & E-core.

I. INTRODUCTION

 Leakage inductance is an important property of transformers in galvanically isolated power electronic converters as it is crucial for the operation of the converter. Galvanically isolated converters can be applied in several applications that enable a more sustainable energy system such as photovoltaic inverters [1], electric traction systems [2], and power quality control of the electric grid [3].

Requirements on transformer leakage inductance depend on the converter topology. In some converter topologies, the leakage inductance should be as small as possible to minimize losses. The flyback converter is an example of these kinds of systems (see [4]). In other topologies, the leakage inductance of the transformer can be exploited as series inductance to replace separate inductors. Thus, the power density is increased and component costs are saved. In these applications, a well-defined leakage inductance is crucial. This concept is frequently applied in resonant converters (see [5]) and dual-active-bridge (DAB) converters [2]. In DAB converters, the series inductance is required for power transfer, power control, and to enable zero voltage switching.

In the design stage of such a converter, the operating and design parameters are usually determined with an optimization procedure, before the components are physically built [6]. Therefore, models of all relevant converter parameters such as transformer leakage inductance, voltages, currents, and losses are required. During the optimization procedure, the models are executed several thousand times (see, e.g., [6]). Consequently, the models have to be computationally efficient to deliver results within a reasonable amount of time.

The transformer leakage inductance can be calculated with numerical methods, reluctance network modeling (RNM), and analytical methods. Numerical methods such as the finite-element method (FEM) [7], [8] are versatile and very accurate but rather time consuming, and therefore, suboptimal for optimizations. RNM [9], [10] is faster, but also less accurate than numerical models. Analytical models such as those proposed by Dowell [11] and Rogowski [12] are usually restricted to their assumed geometry and subject to simplifications, which may reduce the accuracy of the result. However, analytical models are rapidly computable, which makes them very well suited for converter optimizations.

Analytical modeling of the transformer leakage inductance typically consists of the following two steps [11]–[18]:

1) calculate the leakage inductance per unit length \( L' \) from a 2D transformer cross section;
2) scale the per unit length value by the mean length of turns. Recent modeling approaches [19]–[23] pursue a “Double-2D” concept, i.e., the calculation is based on two cross sections of the transformer: inside the transformer window (IW) and outside the transformer window (OW), as depicted in Fig. 1.

These are the two main cross sections that can be deduced from a cut view of E-core and U-core transformers. In the Double-2D approach, the leakage inductance per unit length is
acquired for both cross sections. The per unit length inductances are then scaled by the corresponding partial winding length and summed up to acquire the total leakage inductance. Both of the modeling steps need accurate and fast models to obtain an accurate and computationally efficient overall leakage inductance model. Therefore, it is necessary to investigate both modeling steps separately.

Several reviews and comparisons on leakage inductance calculation have been published. For example, Margueron et al. [15] and Lambert et al. [24] provide a good overview on 2D leakage inductance model approaches. Doeblin et al. [25] compare the accuracy of Rogowski’s model and the mean geometric distance (MGD) model. Fouineau et al. [22] compare the accuracy and calculation times of Dowell’s model [11] and a Double-2D model with three-dimensional (3D) FEM simulations. Kaymak et al. [26] compare a Dowell-based modified leakage inductance model with FEM simulations. However, an extensive comparison of leakage inductance models applied to multiple transformers evaluating computational effort as well as accuracy is still missing.

Previous Double-2D literature [19]–[21] has focused on comparing the overall leakage inductance with measurements and 3D-FEM simulations. This implies the substantial disadvantage that the error cannot be attributed to either of the model steps, or both, i.e., inductance per unit length or scaling length.

Hence, this article elucidates the first step of leakage inductance modeling in detail, i.e., calculating the leakage inductance per unit length. Several leakage inductance per unit length models are compared with respect to accuracy and computational effort. To exclusively evaluate the accuracy of the per unit length models, 2D FEM simulations are taken as reference.

In preceding work [18], the comparison has been executed for the IW cross section. In this article, a separate comparison for the OW cross section is added. These two comparisons allow to attribute the particular error and computational effort to either the model of the IW or OW leakage inductance per unit length. Thus, the best suited model can be selected for each cross section, which is a crucial precondition for accurate and fast Double-2D modeling.

The rest of this article is organized as follows. Section III comprises an overview of leakage inductance modeling steps and elaborates common approaches of leakage inductance per unit length models. Section III introduces the models and compares their modeling approach and concept. Section IV is the key part of this article and compares the models from a practical point of view, i.e., accuracy versus computational effort. Section V elucidates the accuracy of the Double-2D modeling approach compared to the conventional Single-2D modeling by including measurements and 3D FEM simulations. Section VI clarifies the influence of the frequency on the leakage inductance. The Appendices present the general Double-2D and Single-2D equations and the geometrical parameters of the compared transformers. The equations of Roth’s and Margueron’s model are compactly summarized in the external supplementary document of this article.

II. ANALYTICAL LEAKAGE INDUCTANCE MODELING

This section clarifies the most important fundamentals of analytical leakage inductance modeling that are relevant for this article. For an extensive treatise on fundamentals of leakage inductance modeling, see e.g., [27] and [28].

A. Modeling Steps

Analytical Double-2D leakage inductance modeling is typically performed in the following two steps.

1) The leakage inductances per unit length are calculated for IW and OW cross sections: \( L'_{\sigma,in} \) and \( L'_{\sigma,out} \). The values are calculated based on the depicted geometries in Fig. 1.

2) The IW and OW leakage inductances per unit length are multiplied by the respective partial length of the windings \( l_{w,in} \) and \( l_{w,out} \). These are visualized in Fig. 2 for a UR-core and an E-core. Simple equations for \( l_{w,in} \) and \( l_{w,out} \) are proposed in Appendix A.

These shares are finally summed up to the total leakage inductance according to

\[
L_{\sigma,Double-2D} = L'_{\sigma,in} \cdot l_{w,in} + L'_{\sigma,out} \cdot l_{w,out}.
\] (1)

Both model steps introduce error and computational effort into the calculation. Hence, the overall error and computational effort result from both modeling steps. Therefore, it is necessary to analyze both modeling steps separately to understand and identify potential error sources and especially time-consuming parts of the overall model. This article focuses solely on the first step of calculating the IW and OW leakage inductance per
unit length. Thus, the most accurate and efficient calculation models for the particular leakage inductance per unit length can be identified.

B. Method of Images

To obtain accurate IW and OW leakage inductance per unit length values, the presence of the transformer core is taken into account by the method of images. A good treatise on the method of images can be found in [29]. At this point, it is important to clarify the role of the method of images for the OW and IW leakage inductance per unit length.

The OW cross section represents a current next to the center leg, i.e., a ferromagnetic material with finite height and thickness as shown in Fig. 3(a). This is not exactly the constellation that is treated with the method of images. For the ideal mirror constellation, as shown in Fig. 3(b) and (c), the ferromagnetic material has to be of infinite height and thickness. However, Margueron et al. concluded that the thickness of the core is negligible in practical cases with a permeability higher than 100 [16]. In Section IV-B, we will show that also the finite height of the core poses a negligible difference between ideal mirror and OW constellation. Thus, the ideal mirror-constellation represents a very accurate approximation of the OW-constellation and is, therefore, justified to use.

The IW cross section represents currents within the transformer window, resulting in four image planes. This leads to an infinite amount of image layers according to Fig. 4(a). Some models approximate this constellation with a finite amount of image layers. The accuracy loss of this approximation is addressed in Section IV-A.

C. Common Assumptions of Models

In all models, the turns are unified to rectangular winding blocks. The assumed current densities are through-plane \( [J_z(x, y)] \), the resulting magnetic field is in-plane \( [H(x, y), B(x, y)] \), whereas the magnetic vector potential \( A \) only has a through-plane component \( A_z(x, y) \). To account for arbitrary numbers of turns of any winding \( k \), the number of turns \( N_k \) is implied in the current density, i.e., \( J_k = \frac{N_k I_k}{A_{\text{winding},k}} \). where \( J_k \) is the current density of winding \( k \), \( I_k \) is the current through one turn of the winding, and \( A_{\text{winding},k} \) is the surface of the winding \( k \).

The 1D models assume purely axial leakage flux between windings of equal height as shown in Fig. 4(b). The leakage field depends only on the \( x \)-coordinate, but not on the \( y \)-coordinate. Furthermore, the field only has a \( y \)-component \( H_y \). The assumption of purely axial leakage flux neglects the flux fringing at the top and bottom of the windings due to the inevitable isolation distances between windings and yokes as shown in Fig. 4(c).

III. Categorization of Models

The models introduced in this section have been considered because they are applicable to the considered transformer geometries (E, ER, U, UR-core) and sufficiently compact. Roth’s, Margueron’s, and the MGD model are applicable to windings of arbitrary height. Kapp’s, Dowell’s, and Rogowski’s model are only applicable to windings of equal height. Therefore, a winding height transformation is proposed in Appendix D. The considered models are categorized according to Fig. 5 and are shortly summarized as follows.

A. 1D-Models (Axial Leakage Field Assumed, Flux Fringing Neglected)

The 1D models assume purely axial leakage flux between windings of equal height as shown in Fig. 4(b). Thus, the leakage field depends only on the \( x \)-coordinate, but not on the \( y \)-coordinate. Furthermore, the field only has a \( y \)-component \( H_y \). The assumption of purely axial leakage flux neglects the flux fringing at the top and bottom of the windings due to the inevitable isolation distances between windings and yokes as shown in Fig. 4(c).

Kapp [13]: Kapp derived the DC-leakage field \( \vec{H}(x) \) from Ampère’s law assuming axial leakage flux. Next, the magnetic energy density is calculated with \( w_{\text{mag}}(x) = \frac{1}{2} \mu_0 H^2 \), integrated over the transformer window. Finally, the leakage inductance per unit length is derived using (4). Kapp’s model leads to a very simple closed-form formula for the leakage inductance per unit length as follows:

\[
L'_{\sigma,Kapp} = \mu_0 N^2 \left( d + \frac{a_1 + a_2}{3} \right) \frac{1}{h} \quad (2)
\]
where $\mu_0$ is the vacuum permeability, $N$ is the number of turns of the excited winding, and the geometry parameters $a_1$, $d$, $a_2$, and $h$ are defined in Fig. 4(b).

Dowell [11]: Dowell derived the current density $J_2(x, \omega)$ by solving a second-order differential equation derived from Ampère’s law and the law of electromagnetic induction. Next, the voltage $V(\omega)$ across a portion of an arbitrary number of winding layers is derived. The impedance of the transformer is calculated by Ohm’s law. The leakage inductance per unit length is finally derived by $L'_{\sigma} = \frac{2W'_{\text{mag}}}{I^2}$.

B. 2D Models (Flux Fringing Considered)

In 2D models, the flux density $\vec{B}(x, y)$ and the magnetic potential $A_z(x, y)$ are functions of both space coordinates $x$ and $y$. Additionally, the magnetic field has $x$- and $y$-components. Consequently, the flux fringing at the bottom and the top of the windings illustrated in Fig. 4(c) is taken into account. All considered 2D-models assume homogeneous current distribution in the windings. Rogowski’s, Roth’s, and Margueron’s model are based on a solution of the Poisson equation for the magnetic potential $A_z$. These three models compute the magnetic energy per unit length $W'_{\text{mag}}$ according to

$$W'_{\text{mag}} = \frac{1}{2} \int_A A_z \cdot J_z \, dA.$$  \hspace{1cm} (3)

Using the magnetic energy per unit length, the leakage inductance per unit length $L'_{\sigma}$ can be calculated with

$$L'_{\sigma} = \frac{2W'_{\text{mag}}}{I^2}.$$ \hspace{1cm} (4)

where $I$ is the current of the actively excited winding.

Rogowski [12]: Rogowski mirrored the original windings of equal height across the transformer legs. Referring to Fig. 4(a), there is only one horizontal image series in the $x$-direction. The remaining spatial current distribution is expressed as Fourier series dependent on the $x$-coordinate (single-space harmonics). Rogowski’s model is geometrically constrained by the following assumptions:

a) windings of equal height;

b) no gaps between windings and legs [i.e., $d_{x,1} = d_{x,o} = 0$ in Fig. 8(a)];

c) interleaved windings: Constant distance between windings;

d) interleaved windings: Windings of constant width.

With the current distribution, a magnetic potential $A_z(x, y)$ is derived from Poisson’s and Laplace’s equation. Rogowski’s solution consists of several functions valid in either winding domain or air domain and are coupled by intercontinuity boundary conditions. The leakage inductance per unit length $L'_{\sigma}$ is derived using the magnetic energy per unit length $W'_{\text{mag}}$ according to (3) and (4).

Rogowski expressed the infinite Fourier series as a finite expression leading to the same equation that Kapp had derived; only with a correction factor $K$ taking the flux fringing at top and bottom of the windings into account shown as

$$L'_{\sigma, \text{Rogowski}} = K \cdot L'_{\sigma, \text{Kapp}}.$$ \hspace{1cm} (5)

$$K = 1 - \frac{1 - e^{-kh}}{kh} \cdot \left[ 1 - \frac{1}{2} e^{-2kd_{y,b}} (1 - e^{-kh}) \right] \times \left( 1 + e^{-k(d_{y,b} - d_{y,b})} - e^{-k(2d_{y,b} + 2d_{y,b} + h)} \right)$$ \hspace{1cm} (6)

$$k = \frac{\pi}{a_1 + d + a_2}.$$
Roth's model can be simplified by assuming winding-yoke distances $d_{y,b}$ and $d_{y,t}$ to be infinite (see Fig. 8(a) for geometrical parameter definition). In this case, the expression in squared brackets in the Rogowski factor (6) becomes 1 as in [21] and [24]. This model is referred to as “Rogowski simple” in this article.

Roth [14], [31]: Edouard Roth used a double Fourier series dependent of both, $x$- and $y$-coordinates (double space harmonics) to express the current density distribution of each single winding [14]. This leads to the constellation in Fig. 4(a), only with the approximation of $\mu_{r,c} \to \infty$. Next, a potential $A_{z,k}(x,y)$ for each winding $k$ is derived with Poisson’s equation. Unlike Rogowski’s solution, Roth’s potential of each winding $A_{z,k}(x,y)$ is valid for the total computational domain. The total potential $A_z(x,y)$ is calculated by superposing the potentials $A_{z,k}$ of all windings ($A_z = \sum_{k=1}^{C} A_{z,k}$), where $C$ is the number of windings within the transformer window. Analogously to Rogowski, the leakage inductance is derived via the magnetic energy with (3) and (4). Roth’s final equation for the leakage inductance per unit length still contains the double infinite series, however, the series converge rapidly with $\propto \frac{1}{k}$. The equations of Roth’s model are compactly summarized in the supplementary document.

A compact derivation of the leakage inductance from Roth’s potential function was published in [31]. An insightful and compact summary of Roth’s work and a comparison to Rogowski’s approach can be found in [32]. Further publications using Roth’s potential vector can be found in [33] and [34].

Margueron [15], [16]: Margueron’s model is based on the superposition of magnetic potentials of single windings. The method of images is used to mirror these windings one by one. This leads to a discrete amount of image layers for the IW cross section in Fig. 4(a). Margueron’s model is the only considered model that takes the finite permeability of the magnetic core into account.

Margueron derived the potential of each winding $A_{z,k}(x,y)$ from a function commonly used in partial element equivalent circuit (PEEC) modeling. Note that the same potential function has already been derived in 1926 in [35]. A detailed derivation of the potential function can be found in [30, ch. 5.2.2].

Margueron used this potential function to calculate the leakage inductance of transformers. In the 2007 publication [15], the leakage inductance is obtained by numerically integrating the energy density. In the 2010 publication [16], Margueron improved his model by carrying out the integration analytically using a primitive function for the potential $A_{z,k}(x,y)$, which drastically reduces the computational effort. The equations of Margueron’s model are compactly summarized in the supplementary document.

Note that Lambert [24], [36] proposed the same model with a constant offset in the potential (which drops out when calculating the magnetic field), a multiwinding geometry, and an inductance matrix referred to each winding couple. The model in [22] is also mathematically related to Margueron’s model.

Mean Geometric Distances (MGD) [17], [37], [38]: The model simplifies the winding blocks to winding filaments. These winding filaments carry filamentary currents that cause circular magnetic fields. Self-inductance per unit length of the primary winding $L_p'$, self-inductance per unit length of the secondary winding $L_s'$, and mutual inductance per unit length $M'$ of the resulting filamentary coils in air are directly derived from Ampère’s law. The inductances are calculated using geometrically averaged distances between windings (MGD). The core is taken into account by adding mutual inductances per unit length $M_{\text{original-image}}$ between original and discretely mirrored image windings. The core permeability is assumed to be infinite $\mu_{r,c} \to \infty$.

The model was first introduced by J. C. Maxwell in his famous treatise [37, ch. 691] and expanded to leakage inductance by Petrov [17]. The equations of the MGD model can be found in [38].

The aforementioned categorization is illustrated in Fig. 5. Each model has properties that imply advantages and disadvantages. These qualitative properties are compared in Table I.

IV. ACCURACY VERSUS COMPUTATIONAL EFFORT OF MODELS

This section shows the results of the comparison of the leakage inductance per unit length models with respect to the accuracy and computational effort. The model error indicates accuracy, whereas the calculation time indicates computational effort. To exclusively evaluate the accuracy of the per unit length models, the leakage inductance per unit length calculated by 2D FEM is taken as reference to determine the error. The models are applied to the geometries of six transformer prototypes listed in Table V. A comparison is presented for both, IW and OW cross section.

A. Inside-Window (IW)

Fig. 6(a) and (b) shows the model error (accuracy) versus the calculation time (computational effort) of the compared models applied to the particular IW geometries of the compared transformers in Table V.

Roth’s model is the most accurate in every considered geometry. The error tends toward 0%, as Roth’s model represents the full analytical solution to the originally posed field problem. Roth’s model takes both, the spatially distributed current density and the magnetic core (with $\mu_{r,\text{core}} \to \infty$) into account. The computational effort of the model is remarkably small considering the calculation times of around 0.2 ms. As the double Fourier series converges rapidly ($\propto \frac{1}{k}$), the amount of sum terms in the horizontal and vertical Fourier series are set to 25. These values can be increased/decreased to increase/decrease the accuracy resulting in higher/lower computational effort.

Rogowski’s model is a viable solution for optimizations when it comes to geometries with rather small winding-leg distances. Transformers No.1–3 are exemplary for this, as the error is below 1% in these scenarios. The small computational effort of Rogowski’s formula is the biggest advantage of the model, which is due to the compactness of the model’s closed-form leakage inductance formula (5). Calculation times are as low as approximately 5–10 μs.

Margueron’s model yields below 5% error in all considered geometries, which makes the model versatile in geometry. Note
that the analytical integration of the energy integral according to the Margueron 2010 model [16] decreases the computational effort of the model significantly by about three orders of magnitude. The Margueron 2007 model [15] is much slower because values of the magnetic potential all over the windings need to be calculated explicitly to numerically compute the energy integral (3). Margueron’s model has an important advantage over the other models: The finite magnetic permeability of the core \( \mu_r, c \) is taken into account. This is essential, when it comes to low permeability cores.

For this comparison, only one image layer was used. The accuracy of Margueron’s model can be further increased by adding more layers of images by the cost of higher calculation time.

The MGD model is the second fastest model and yields errors below 3% in geometries with similar winding dimensions (i.e., \( a_1 \approx a_2 \), resp., \( h_1 \approx h_2 \)). Transformers No.1 and 4–6 are examples for this. The MGD model is not suitable for geometries with high relative height differences \( h_{1,rel} = \frac{h_1}{h_2} \), in which the error becomes relatively high (No.3: 9%, No.2: 6%). This is due the fact that the approximation of circular fields is not accurate in this case. This circumstance makes the model quite unreliable as the error very much depends on the geometry. Similar to Margueron’s model, one image layer was specified for this comparison and the accuracy of the MGD model can be increased by adding more layers of images by the cost of higher computational effort.

**Kapp’s model** is the most inaccurate because it neglects the nonaxial part of the leakage field between the windings, i.e., the flux fringing effect. The error reaches 16% in the most critical case (tr. No. 5). The flux fringing effect is mainly caused by the practically unavoidable isolation distances between windings and core [geometry parameters \( d_{y,b} \) and \( d_{y,t} \) in Fig. 8(a)]. Kapp’s model is the model with the least computational effort due to its very compact closed formula (2). The computation time is around 1–2 \( \mu \)s.

**Dowell’s model** yields the same result as Kapp’s model for the considered static leakage inductance. Therefore, Dowell’s model is equally inaccurate as Kapp’s model due to the neglected flux fringing. The computational effort is remarkably high due to the consideration of frequency. As the magnetostatic inductance is already erroneous due to the 1D-approximation of the field, the consideration of the frequency does not yield great benefits.

**B. Outside Window (OW)**

First, proof is required that the approximation of applying an ideal mirror [core with infinite height and thickness, Fig. 3(b) and (c)] to the OW constellation [core with finite height and thickness, Fig. 3(a)] is justified, as elaborated in Section II-B.
To prove that the introduced error of finite height and finite thickness of the core material is negligibly small for the considered transformers, FEM simulations have been performed. These show that the difference in leakage inductance between the ideal mirror-constellation and the OW-constellation is negligibly small. Table II shows the error introduced by this approximation for all considered transformers.

Fig. 6(c) and (d) shows the error versus the calculation time of the models applied to the considered transformer geometries for the OW cross section. Fewer models are compared for the OW cross section because Kapp’s model, Dowell’s model, and Rogowski’s model are not applicable to this cross section.

Margueron’s model is very accurate with an error range of 0%–0.2%. This error is only caused by approximating the OW constellation [see Fig. 3(a)] with the ideal mirror constellation [see Fig. 3(b) and (c)]. Mathematically, Margueron’s model represents the full analytical solution to the ideal-mirror constellation in Fig. 3(b) and (c). This can be seen in Table III that shows the errors of the leakage inductances per unit length calculated with the Margueron 2010 model and the ideal mirror calculated by 2D FEM, which are very close to 0%.

Margueron’s model is accurate in the OW cross section because Margueron’s solution fully takes the core with finite
permeability and the spatially distributed current distribution into account.

Note that the Margueron 2010 model [16] is much faster than the Margueron 2007 model [15]. This is due to the explicit field calculation required for the version of 2007 (just as for the IW cross section). The Margueron 2010 model requires approximately 0.2 ms of calculation time.

Roth’s model is also quite accurate in the OW cross section. The error is only slightly higher in comparison to Margueron’s model. However, the computational effort exceeds the Margueron 2010 model, making it a worse choice than the Margueron 2010 model.

Roth’s model is tailored to an IW cross section. The trick to apply Roth’s model to the OW cross section is to set the yokes and the outer leg sufficiently far away from the windings. In the particular case, the distances \( d_{y,b} \), \( d_{y,t} \), and \( d_{x,o} \) [see Fig. 8(a)] have been set to \( 10 \frac{L_0}{2} \). This parametrization was empirically found to yield good results.

Note that higher values of \( u_w \) and \( h_a \) hamper the convergence of the double Fourier series in Roth’s model. This results in a higher number of sum terms in both of the Fourier series required for achieving satisfactory accuracy. For the OW cross section, 250 sum terms are specified in horizontal and vertical Fourier series, which naturally increases the calculation time.

The MGD model is the fastest of the compared models. However, the error depends very much on the geometry, as can be seen from the transformer No.2 (error \( \approx 5\% \)) and transformer No.3 (error \( \approx 8\% \)). Just as for the IW cross section, this is due to the considerable relative height difference \( \frac{h_1 - h_2}{h_{mean}} \) and the approximation of using circular fields.

Further details concerning the time extraction and calculation routine can be found in [18, Sec. 3.1]. Each model relies on specific simplifications that inevitably introduce error into the calculation. The errors of the particular models were attributed to geometrical circumstances and model simplifications in [18, Sec. 3.2].

V. ACCURACY OF DOUBLE-2D MODELING

The main focus of this article is to identify the fastest and most accurate leakage inductance per unit length models for IW and OW cross sections. However, leakage inductance per unit length values cannot be compared to measurements as there is no way to measure them. Therefore, the scaling length models proposed in Appendix A are used to obtain the total calculated leakage inductance. These total leakage inductances can be compared to the measured transformer leakage inductance.

To assess the accuracy potential of the Double-2D approach, the static leakage inductance of each transformer is measured. Furthermore, the leakage inductances per unit length calculated by Roth’s model and Dowell’s model are scaled using the Single-2D approach (7) as well as the Double-2D approach (1). Furthermore, 3D-FEM simulations are conducted. Table IV shows the measured leakage inductances and the error of the particular models applied to each of the compared transformers. Moreover, the normalized root mean square error (nRMSE) of each model is given. The nRMSE indicates the rough scale of error independently of the transformer geometry.

The Double-2D modeling approach (1) yields considerably lower error than the Single-2D modeling approach (7). This can be observed when comparing the nRMSE values of Roth-Double-2D (8.5\%) and Roth-Single-2D (13.9\%). Roth’s leakage inductance per unit length model is selected for assessing the accuracy of Double-2D compared to Single-2D, as Section IV showed that Roth’s model yields errors close to 0\% for IW and OW cross section.

Section IV already showed that 2D leakage inductance per unit length models are significantly more accurate than 1D leakage inductance per unit length models. This error is also reflected in the overall leakage inductance. This conclusion can be drawn from comparing Roth’s Single-2D model to Dowell’s Single-2D model. Here, the nRMSE is 13.9\% and 24.6\%, respectively.

It has to be clarified that obtaining 0\% error is practically not possible with any model. The main reason is that geometrical and material properties will never be exactly the same as specified in the model. Other reasons are unavoidable measurement inaccuracies and measurement influences. This circumstance can be observed when looking at the 3D-FEM error. With equal geometry parameters of the model and real transformer, the measured value and 3D-FEM value should coincide very well as 3D-FEM is very accurate. However, the error is in a range of up to approximately 5\% with an nRMSE of 3.3\%. This is roughly the
of the transformer No. 2 (see Table V) in the relevant frequency range.

**VI. FREQUENCY DEPENDENCE OF LEAKAGE INDUCTANCE**

Galvanically isolated power converters operate in the medium frequency range—typically at a few tens of kilohertz (e.g., [1] and [2]). At these frequencies, skin and proximity effect play a role in the redistribution of the current density within a conductor and must be considered for winding resistance calculations. However, leakage inductance depends on the stored magnetic energy—which is a global physical quantity. As global quantity, magnetic energy is not heavily affected by local current density redistributions within a conductor because the conductor cross section is designed to be only a small fraction of the total winding cross section in a technical transformer. In other words, local current density redistributions within a proportionally small conductor represent only a minor macroscopical fluctuation. Therefore, the effect of frequency on leakage inductance is not significant for technical transformers within the relevant frequency range.

To prove this statement, the short-circuit inductances of the transformers were measured statically and as a function of frequency. Fig. 7 shows the measured static short-circuit inductance $L_{sc,0}$ and the frequency-dependent short-circuit inductance $L_{sc}(\omega)$ of the transformer No. 2 (see Table V) in the relevant frequency range of 10 kHz up to 1 MHz. Both windings of the transformer No.2 are foil windings. These are typically heavier and thicker than Litz wire windings due to their considerable conductor height. Therefore, the transformer No.2 reflects a worst-case scenario when it comes to frequency dependence. Still, the figure shows that the difference in leakage inductance due to frequency effects is as low as 4% in the relevant frequency range.

As frequency effects play a minor role in leakage inductance, only one frequency-dependent model (Dowell’s model [11]) was considered in the comparison. The big disadvantage with Dowell-based 1D models is that their solution of the magnetic field is one-dimensional, i.e., the leakage flux is assumed purely axial as shown in Fig. 4(b). Section IV showed that this assumption of neglecting the fringing flux introduces significantly higher error than the error introduced by frequency effects. Additionally, 1D models are not applicable to windings of different height, which is another substantial disadvantage. As a consequence, the conducted comparison is focused on static 2D models instead of frequency-dependent 1D models due to their consideration of fringing leakage flux and higher geometrical versatility.

**VII. CONCLUSION**

Several analytical 1D and 2D leakage inductance per unit length models have been compared and assessed with respect to a tradeoff between accuracy and computational effort. The comparison has been executed for both, IW and OW cross section. Depending on the requirements on accuracy and computation time, the best suited model can be chosen for IW and OW leakage inductance per unit length.

Regarding the IW cross section, Roth’s model delivers the best accuracy combined with rather low computational effort. The error referred to the 2D FEM simulation is negligibly small as the model represents the complete analytical solution to the posed leakage inductance problem. The calculation times are as low as 0.2 ms on a standard notebook. The model is geometrically versatile as it is applicable to all geometries with rectangular windings within a rectangular transformer window, given that winding and window edges are parallel.

As long as the distances between windings and core are small compared to the transformer window dimensions, Rogowski’s model yields satisfactory accuracy (less than 1% in the simulated cases). Rogowski’s closed-form leakage inductance formula is rapidly executable with calculation times around 5–10 $\mu$s.

The model of Margueron is the only considered model that takes the finite permeability of the core $\mu_r \mu_{core}$ into account. Transformers with a low-permeability core are, therefore, best examined with this model. Furthermore, the error is below 5% in all simulated transformer geometries. The accuracy of the model can be further improved by increasing the number of image layers (by the cost of additional computational effort). These properties make Margueron’s model the most versatile.

The 1D-models such as Dowell’s and Kapp’s model result in considerably large errors up to 16% for the leakage inductance per unit length as they neglect the flux fringing caused by the distances between windings and core yokes.

Regarding the OW cross section, Margueron’s model is the best performing model. Here, the error is below 0.2% in all simulated cases. It was shown that the remaining error does not result from the model itself but from the simplification of applying an ideal mirror (center leg with infinite height and thickness) to the OW cross section (center leg with finite height and thickness). The computation time of Margueron’s model applied to the OW cross section is approximately 0.2 ms on a standard notebook.
TABLE V
PARAMETERS OF CONSIDERED TRANSFORMERS, SEE Fig. 8 FOR PARAMETER DEFINITIONS

<table>
<thead>
<tr>
<th>Transformer geometry parameters (mm)</th>
<th>Further data</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁, a₂, d, h₁, h₂, h₃, dₐ, dₐ, dₓ₁, dₓ₂, bₙₑg, αₙₑg, dₙₑg</td>
<td>Core, C. leg, N₁, N₂, Wind.1, Wind.2</td>
</tr>
<tr>
<td><strong>No.1</strong> 4.9 4.9 17.5 90.0 80.0 5.0 3.0 3.0 0.0 7.0 56.0 30.0</td>
<td>E rect. 20 20 Foil Foil</td>
</tr>
<tr>
<td><strong>No.2</strong> 4.2 4.6 4.0 52.0 44.0 4.0 1.9 1.9 0.0 7.2 20.2 20.2</td>
<td>E rect. 23 26 Foil Foil</td>
</tr>
<tr>
<td><strong>No.3</strong> 3.8 4.5 7.0 48.9 37.9 5.5 1.5 1.5 0.0 8.5 26.0 28.1</td>
<td>E rect. 19 18 Foil Foil</td>
</tr>
<tr>
<td><strong>No.4</strong> 4.1 4.3 7.7 72.0 72.0 0.0 16.0 13.0 4.0 10.9</td>
<td>- 26.9</td>
</tr>
<tr>
<td><strong>No.5</strong> 2.5 12.0 14.0 72.0 72.0 0.0 12.2 12.2 3.5 15.5</td>
<td>- 37.5</td>
</tr>
<tr>
<td><strong>No.6</strong> 3.0 6.5 5.0 40.0 37.0 1.5 4.5 6.5 2.0 3.5</td>
<td>- 23.4</td>
</tr>
</tbody>
</table>

APPENDIX A
PARTIAL LENGTH OF WINDINGS EQUATIONS

For conventional Single-2D modeling, the leakage inductance per unit length of the IW cross section \( L'_{\sigma, \text{in}} \) is multiplied by the mean length of turns \( l_m \) according to

\[
L_{\sigma, \text{Single-2D}} = L'_{\sigma, \text{in}} \cdot l_m.
\]  
(7)

The mean length of turns \( l_m \) and partial length of the windings \( l_{\text{w, in}} \) and \( l_{\text{w, out}} \) depend on the shape of the center leg. In case of a circular center leg, the lengths are curved, in case of a rectangular leg, the lengths are straight. To take the core shape into account, the factor \( s_{\text{core}} \) is introduced according to

\[
s_{\text{core}} = 1 \ldots \text{for U- and UR-core} \\
s_{\text{core}} = 2 \ldots \text{for E- and ER-core}.
\]  
(8)

The mean length of turns \( l_m \) is not uniquely defined in the literature. In this article, the definition according to Hurley [28] is used.

**A. Circular Center Leg**

According to [28], the mean length of turns \( l_m \) is calculated according to

\[
l_m = 2\pi \left( \frac{r_1 + r_2}{2} \right) = 2\pi (d_{\text{leg}} + 2d_{x,i} + a_1 + d + a_2).
\]  
(9)

where \( r_1 \) is the inner diameter of the inner winding and \( r_2 \) is the outer diameter of the outer winding.

To perform Double-2D modeling, the IW and OW partial lengths of the windings \( l_{\text{w, in}} \) and \( l_{\text{w, out}} \) are required. The proposed equations are as follows:

\[
l_{\text{w, in}} = s_{\text{core}} \cdot \frac{\alpha}{2\pi} \cdot l_m
\]  
(10)

\[
l_{\text{w, out}} = \frac{2\pi - s_{\text{core}} \cdot \alpha}{2\pi} \cdot l_m.
\]  
(11)

The angle \( \alpha = 2 \arcsin \left( \frac{d}{d_{\text{leg}} + 2d_{x,i} + 2a_1 + 2d + 2a_2} \right) \) represents the share of the length that belongs to the IW cross section [see Fig. 8(c)].

**B. Rectangular Center Leg**

The mean length of windings in case of a rectangular center leg can be calculated according to

\[
l_m = 2b_{\text{leg}} + 2a_{\text{leg}} + 4(d_{x,i} + a_1 + d + a_2).
\]  
(12)

APPENDIX B
DEFINITION OF GEOMETRICAL PARAMETERS

The geometrical parameters of an arbitrary transformer are illustrated in Fig. 8.

APPENDIX C
PARAMETERS OF COMPARED TRANSFORMERS

The geometrical parameters of the six compared transformers are listed in Table V.

APPENDIX D
WINDING HEIGHT TRANSFORMATION (REQUIRED FOR KAPP, DOWELL, AND ROGOWSKI)

For transforming windings of different height into windings of equal height, it is proposed to proceed as following: The geometric mean height is calculated according to (15). The geometric mean was found to yield more accurate results than the arithmetic mean. The distances from windings to yokes \( d_{y,b} \) and \( d_{y,t} \) are corrected according to (16), such that the window height \( h_w \) is the same as before

\[
h_{\text{geom}} = h = \sqrt{h_1 \cdot h_2}
\]  
(15)

\[
d_{y,b,\text{corr}} = d_{y,b} + \frac{h_1 - h}{2}; \quad d_{y,t,\text{corr}} = d_{y,t} + \frac{h_1 - h}{2}.
\]  
(16)
REFERENCES


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Abstract—This supplementary document provides the equations of Roth’s and Margueron’s model in a compact and comprehensible way. The parameter definitions concur with the definitions of the main paper in Fig. 8a.

Equations of Roth’s Model [14], [31]

Applying the image method to the inside-window cross section leads to a configuration according to Fig. 4a. Roth assumes infinite permeability of the core \( \mu_{k,c} \rightarrow \infty \). Hence, the image current densities are equal to the original current density due to \( J_{\text{image}} = J_{\text{original}} \frac{\mu_{k,c} - 1}{\mu_{k,c} + 1} \). Thus, the \( z \)-component of the current density distribution of each winding \( k \) \( J_{z,k}(x, y) \) becomes a two-dimensional rectangular signal in space that can be expressed as double Fourier series according to (17)

\[
J_{z,k}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} J_{k,m,n} \cos \left( \frac{m \pi}{T_x} x \right) \cos \left( \frac{n \pi}{T_y} y \right)
\]

(17)

where \( J_{k,m,n} \) are the Fourier coefficients and \( T_x \) and \( T_y \) are the periods in \( x \)- and \( y \)-direction.

Fig. 9 shows the current distribution in space. The figure shows that the periods of the signal in \( x \)- and \( y \)-direction can be expressed according to (18)

\[
T_x = 2w_w \quad T_y = 2h_w
\]

(18)

where \( w_w \) is the window width and \( h_w \) is the window height. This leads to the Fourier series in (19)

\[
J_{z,k}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} J_{k,m,n} \cos \left( \frac{m \pi}{w_w} x \right) \cos \left( \frac{n \pi}{h_w} y \right)
\]

(19)

Next, the Fourier coefficients have to be calculated according to (20). The Fourier coefficients vary depending on whether \( m \) or \( n \), or both equal zero. Note that the even symmetry of the function is exploited to shorten the integrals

\[
J_{k,00} = \frac{2}{T_x T_y} \int_{0}^{T_y/2} \int_{0}^{T_x/2} J_k \, dx \, dy = \frac{1}{w_w h_w} \int_{h_w}^{h_w} \int_{a_k}^{a_k} J_k \, dx \, dy
\]

\[
J_{k,m0} = \frac{4}{T_x T_y} \int_{0}^{T_y/2} \int_{0}^{T_x/2} J_k \cos \left( \frac{m \pi}{w_w} x \right) \, dx \, dy
\]

\[
= \frac{2}{w_w h_w} \int_{h_w}^{h_w} \int_{a_k}^{a_k} J_k \cos \left( \frac{m \pi}{w_w} x \right) \, dx \, dy
\]

\[
J_{k,0n} = \frac{2}{T_x T_y} \int_{0}^{T_y/2} \int_{0}^{T_x/2} J_k \cos \left( \frac{n \pi}{h_w} y \right) \, dx \, dy
\]

\[
= \frac{2}{w_w h_w} \int_{h_w}^{h_w} \int_{a_k}^{a_k} J_k \cos \left( \frac{n \pi}{h_w} y \right) \, dx \, dy
\]

\[
J_{k,m,n} = \frac{4}{T_x T_y} \int_{0}^{T_y/2} \int_{0}^{T_x/2} J_k \cos \left( \frac{m \pi}{w_w} x \right) \cos \left( \frac{n \pi}{h_w} y \right) \, dx \, dy
\]

\[
= \frac{4}{w_w h_w} \int_{h_w}^{h_w} \int_{a_k}^{a_k} J_k \cos \left( \frac{m \pi}{w_w} x \right) \cos \left( \frac{n \pi}{h_w} y \right) \, dx \, dy
\]

(20)
according to (27)

The potentials of each winding are superposed according to the current density, resulting in (30)

Next, equations (27) and (30) are set into the Poisson equation for the magnetic potential in 2D cartesian coordinates $A_z(x, y)$

This delivers the relation between the Fourier coefficients of the magnetic potential and the current density according to (32)

Next, the magnetic energy per unit length $W'_\text{mag}$ is acquired.

Note that the Fourier coefficient $J_{00} = 0$ does not contribute to the magnetic energy due the balanced magnetomotive force (MMF) of primary and secondary side ($MMF_1 = N_1I_1 = -MMF_2 = -N_2I_2$). The product $A_z(x, y) \cdot J_z(x, y)$ does not yield mixd sum terms due to the orthogonality relation (34).

where $T$ is the period of the signal and $m$ and $n$ are arbitrary integers.

The $\cos^2$ functions were integrated using (35). This integration is essential for computational efficiency, as the constant primitive function spares the explicit calculation of the field inside the windings.

Finally, the leakage inductance per unit length $L'_\alpha$ is obtained
The transformer core is taken into account by discrete images for Margueron’s model

\[ L'_\sigma = \frac{2 W'_\text{mag}}{I_{\text{ref}}^2} \]  

(36)

where \( I_{\text{ref}} \) is the current of the side the inductance is referred to (primary or secondary).

The series in (33) are very fast convergent \( \propto \frac{1}{r^4} \). However, higher values of \( w_w \) and \( h_w \) hamper the convergence of the series. The results of this paper were produced with a number of series terms of 25 (inside-window), resp. 250 (outside-window). For the outside-window cross section, a higher number of sum terms is required due to the very high values of \( w_w \) and \( h_w \).

**Equations of Margueron’s Model** [15], [16]

Margueron’s model is based on a formula for the vector potential in space due to a current-carrying winding \( k \) embedded in a homogeneous medium with the permeability \( \mu_0 \) as a function of its location and dimensions. The \( z \)-component of the vector potential \( A_{z,k}(x, y) \) due to the current in winding \( k \) can be calculated according to (37)

\[
A_{z,k}(x, y) = \frac{-\mu_0}{4\pi} J_k \left[ F \left( x - x_{c,k} - \frac{w_k}{2}, y - y_{c,k} - \frac{h_k}{2} \right) \right. \\
- F \left( x - x_{c,k} + \frac{w_k}{2}, y - y_{c,k} - \frac{h_k}{2} \right) \\
- F \left( x - x_{c,k} - \frac{w_k}{2}, y - y_{c,k} + \frac{h_k}{2} \right) \\
+ F \left( x - x_{c,k} + \frac{w_k}{2}, y - y_{c,k} + \frac{h_k}{2} \right) \right]
\]

(37)

where \( J_k = \frac{N_k i_k}{w_k h_k} \) is the homogeneous current density of winding \( k \). Just as in Roth’s model, the number of turns \( N_k \) is already implied in the current density. The variables \( x_{c,k} \) and \( y_{c,k} \) are the coordinates of the center of the winding, \( w_k \) is the width of the winding, and \( h_k \) is the height of the winding, as shown in Fig. 10. The function \( F(X, Y) \) can be calculated according to (38)

\[
F(X, Y) = XY \ln(X^2 + Y^2) \\
+ X^2 \arctan \left( \frac{Y}{X} \right) + Y^2 \arctan \left( \frac{X}{Y} \right)
\]

(38)

The singularities at \( X = 0 \) and \( Y = 0 \) are avoided by setting the respective \( \arctan \)-term to 0, which is justified due to the prefactor \( X^2 \) resp. \( Y^2 \).

Fig. 10: Relevant geometry parameters of winding \( k \) required for Margueron’s model

The transformer core is taken into account by discrete images of the windings. In the outside-window cross section, the windings are mirrored across the edge of the center transformer leg according to section II-B of the main document. In the inside-window cross section, the windings are mirrored according to Fig. 4a, only with a discrete amount of image layers. The amount of image layers has to be assumed. Assuming one image layer is the fastest and the least accurate way. Adding image layers leads to higher accuracy but also higher computational effort. Note that the finite permeability of the core \( \mu_{\text{ref}} \) is taken into account by scaling the mirrored currents according to \( J_{\text{image}} = J_{\text{original}} \frac{\mu_{r,c}}{\mu_{\text{ref}}} \).

Margueron’s model also uses the magnetic energy integral (39).

\[
W'_\text{mag} = \frac{1}{2} \int_{\text{window}} A_{z,\text{tot}} \cdot J_z \, dA
\]

(39)

The total potential \( A_{z,\text{tot}}(x, y) \) is a superposition of the potential due to all original and mirrored currents according to (40)

\[
A_{z,\text{tot}}(x, y) = \sum_{k=1}^{D} A_{z,k}(x, y)
\]

(40)

where \( D \) is the amount of all windings, including the image windings. The number \( D \) depends on the number of images, i.e. the particular cross section and the amount of image layers. Note that the integral (39) is only nonzero in the winding-domain since the rest of the space is without current.

**Margueron 2007-model** [15]

In the Margueron 2007-model [15], the integration in (39) is executed numerically. I.e. several points in space are selected and the values are calculated. Between these points, the potential is linearly interpolated. The points without current can be neglected, since they don’t contribute to the integral (39).

**Margueron 2010-model** [16]

In the Margueron 2010-model [16], the integral (39) is executed analytically. The contribution to the magnetic energy of each original and mirrored conductor \( k \) is separately calculated. This is done by computing (41)

\[
W'_{\text{cont}, k} = \int_{\text{window}} A_{z,k} \cdot J_z \, dA
\]

(41)

Assuming a 2-winding transformer as depicted in Fig. 8a (indices 1 resp. 2), (41) splits up in two integrals

\[
W'_{\text{cont}, k} = J_1 \int_{\text{conductor 1}} A_k(x, y) \, dx \, dy
+ J_2 \int_{\text{conductor 2}} A_k(x, y) \, dx \, dy
\]

(42)
where $J_1$ and $J_2$ are the current densities of conductor 1 and conductor 2, respectively.

The integrals in (42) can be solved analytically using (43)

$$
\int_{\text{conductor } i} F(x - x_{c,k} \pm \frac{w_k}{2}, y - y_{c,k} \pm \frac{h_k}{2}) \, dA =
$$

$$
= \int_{h_i^-}^{h_i^+} \int_{a_i^-}^{a_i^+} F(x - x_{c,k} \pm \frac{w_k}{2}, y - y_{c,k} \pm \frac{h_k}{2}) \, dx \, dy =
$$

$$
= G(a_i^- - x_{c,k} \pm \frac{w_k}{2}, h_i^- - y_{c,k} \pm \frac{h_k}{2}) - G(a_i^- - x_{c,k} \pm \frac{w_k}{2}, h_i^- - y_{c,k} \pm \frac{h_k}{2}) - G(a_i^- - x_{c,k} \pm \frac{w_k}{2}, h_i^+ - y_{c,k} \pm \frac{h_k}{2}) + G(a_i^- - x_{c,k} \pm \frac{w_k}{2}, h_i^+ - y_{c,k} \pm \frac{h_k}{2})
$$

where the index $i$ represents the particular original conductor (range of integration) and the index $k$ represents the conductor that causes the field.

The primitive function $G(X, Y)$ is given in (44)

$$
G(X, Y) = -\frac{1}{24} (X^4 - 6X^2Y^2 + Y^4) \ln(X^2 + Y^2)
$$

$$
+ \frac{1}{3} XY \left[ X^2 \arctan \left( \frac{Y}{X} \right) + Y^2 \arctan \left( \frac{X}{Y} \right) \right]
$$

$$
- \frac{7}{24} X^2Y^2
$$

(44)

Again, the singularities at $X = 0$ and $Y = 0$ are avoided by setting the respective arctan-term to 0, which is justified due to the prefactor $X^2$ resp. $Y^2$.

Next, the contributions to the magnetic energy of all conductors are superposed.

$$
W'_{\text{mag, tot}} = \sum_{k=1}^{D} W'_{\text{cont}, k}
$$

(45)

where $D$ indicates the amount of all conductors, i.e. original and mirrored conductors. Finally, the leakage inductance per unit length $L'_\sigma$ can be computed by (46)

$$
L'_\sigma = \frac{2W'_{\text{mag, tot}}}{I_{\text{ref}}^2}
$$

(46)

where $I_{\text{ref}}$ is the current of the side the inductance is referred to (primary or secondary).